AUSLANDER-REGULAR AND COHEN-MACAULAY QUANTUM GROUPS

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Let $U_q(C)$ be the quantum group or quantized enveloping algebra in the sense of [6, 7] associated to a Cartan matrix C. A relevant property of $U_q(C)$ is that it can be endowed with a multi-filtration such that the associated multi-graded algebra is an easy localization of the coordinate ring of a quantum affine space [7, Proposition 10.1]. Thus, it is not surprising if we claim that $U_q(C)$ is an Auslander-regular and Cohen-Macaulay algebra (see, e.g., [2] for these notions). However, when one tries to construct a mathematically sound argument to prove this, one realizes that there are not ready-to-use results for this in the literature. Here we use re-filtering methods (see Theorem 1) similar to that in [5] and [4] to prove, in conjunction with results from [2] and [14], that certain types of multi-filtered algebras are Auslander-regular and Cohen-Macaulay (Theorem 3). This is applied to obtain that $U_q(C)$ is Auslander-regular and Cohen-Macaulay.

In this note, K denotes a commutative ring and $\mathbb{N}^n$ is the free abelian monoid with n generators $\epsilon_1, \dots, \epsilon_n$. The elements in $\mathbb{N}^n$ are vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ with non-negative integer entries. An admissible order $\preceq$ on $\mathbb{N}^n$ is a total order compatible with the sum in $\mathbb{N}^n$ and such that $0 \preceq \alpha$ for every $\alpha \in \mathbb{N}^n$. In this way, $\mathbb{N}^n$ becomes a well-ordered monoid. A fundamental example of admissible order on $\mathbb{N}^n$ is the lexicographical order $\leq_{lex}$ with $\epsilon_1 <_{lex} \cdots <_{lex} \epsilon_n$. Every vector $\mathbf{w}$ with strictly positive entries gives an example of admissible order $\preceq_{\mathbf{w}}$ by putting

$$\alpha \preceq_{\mathbf{w}} \beta \iff \begin{cases} \langle \mathbf{w}, \alpha \rangle < \langle \mathbf{w}, \beta \rangle & \text{or} \\ \langle \mathbf{w}, \alpha \rangle = \langle \mathbf{w}, \beta \rangle \quad \text{and} \quad \alpha \leq_{\text{lex}} \beta \end{cases} \tag{1}$$

where $\langle -, - \rangle$ denotes the usual dot product in $\mathbb{R}^n$.

An $(\mathbb{N}^n, \preceq)$–filtration on a K–algebra R is a family $F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$ of K–submodules of R such that

1. $F_\alpha(R) \subseteq F_\beta(R)$ for all $\alpha \preceq \beta \in \mathbb{N}^n$.
2. $F_\alpha(R)F_\beta(R) \subseteq F_{\alpha+\beta}(R)$ for all $\alpha, \beta \in \mathbb{N}^n$.
3. $\bigcup_{\alpha \in \mathbb{N}^n} F_\alpha(R) = R$.
4. $1 \in F_0(R)$.

The associated $\mathbb{N}^n$–graded algebra is given by $G^F(R) = \bigoplus_{\alpha \in \mathbb{N}^n} G^F_\alpha(R)$, where $G^F_\alpha(R) = F_\alpha(R)/F^-_\alpha(R)$ and $F^-_\alpha(R) = \bigcup_{\beta \prec \alpha} F_\beta(R)$. Further details can be found in [9]. The multi-degree of a nonzero element $r \in R$ is defined as $\operatorname{mdeg}(r) = \min\{\alpha \in \mathbb{N}^n \mid r \in F_\alpha(R)\}$.

When $n = 1$, the only admissible order is the usual one and multi-filtrations are just positive filtrations. In this case, the associated graded algebra will be denoted by $\mathrm{gr}(R)$.

We will use extensively the following terminology: Let Λ be a subalgebra of an algebra R, and let $x_1, \dots, x_n$ be elements in R. A standard monomial in $x_1, \dots, x_n$ is an expression $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Assume that an element $r \in R$ can be written in the form

$$r = \sum_{\alpha \in \mathbb{N}^n} r_\alpha \mathbf{x}^\alpha \qquad (r_\alpha \in \Lambda) \tag{2}$$

The expression (2) is called a (left) standard representation of r. We will often refer as (left) polynomials to the elements of R having a standard representation.

Theorem 1. *Let Λ be a left noetherian subalgebra of a K–algebra R, let s be a positive integer and let $q_{ji} \in \Lambda$ for $1 \leqslant i < j \leqslant s$. The following statements are equivalent*

(i) There is an admissible order $\preceq$ on some $\mathbb{N}^n$ and an $(\mathbb{N}^n, \preceq)$–filtration $F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$ on R such that $F_0(R) = \Lambda$, every $F_\alpha(R)$ is finitely generated as a left Λ–module and $G^F(R) = \Lambda[y_1; \sigma_1] \dots [y_s; \sigma_s]$ is an $\mathbb{N}^n$–graded iterated Ore extension for some homogeneous elements $y_1, \dots, y_s$ such that $\sigma_j(y_i) = q_{ji} y_i$ for every $1 \leqslant i < j \leqslant s$.

(ii) There is an $\mathbb{N}$–filtration $\{R_n \mid n \in \mathbb{N}\}$ on R such that $R_0 = \Lambda$, every R_n is finitely generated as a left Λ–module and $\mathrm{gr}(R) = \Lambda[y_1; \sigma_1] \dots [y_s; \sigma_s]$ is an $\mathbb{N}$–graded iterated Ore extension for some homogeneous elements $y_1, \dots, y_s$ such that $\sigma_j(y_i) = q_{ji} y_i$ for every $1 \leqslant i < j \leqslant s$.

(iii) There are elements $x_1, \dots, x_s \in R$, an admissible order $\preceq'$ on $\mathbb{N}^s$, and finite subsets $\Gamma_{ji}, \Gamma_k \subseteq \mathbb{N}^s$ for $1 \leqslant i < j \leqslant s, 1 \leqslant k \leqslant s$ with $\max_{\preceq'} \Gamma_{ji} \prec' \epsilon_i + \epsilon_j$ and $\max_{\preceq'} \Gamma_k \prec' \epsilon_k$ such that $\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{N}^s\}$ is a basis of R as a left Λ–module and $x_j x_i = q_{ji} x_i x_j + \sum_{\alpha \in \Gamma_{ji}} c_\alpha \mathbf{x}^\alpha$ and for all $a \in \Lambda$, $x_k a = a^{(k)} x_k + \sum_{\alpha \in \Gamma_i} c_\alpha \mathbf{x}^\alpha$.

Proof. (i) implies (iii). Let $\alpha_i \in \mathbb{N}^n$ denote the multi-degree of y_i for $1 \leqslant i \leqslant s$. Clearly, $\{\mathbf{y}^\gamma \mid \gamma \in \mathbb{N}^s\}$ is a basis of $G^F(R)$ as a left Λ–module. Thus, given $r \in R$, the homogeneous element $r + F^-_{\mathrm{mdeg}(r)}(R) \in G^F(R)$ has a unique representation as homogeneous standard left polynomial in $y_1, \dots, y_s$ with coefficients in Λ. Thus,

$$r + F^-_{\mathrm{mdeg}(r)}(R) = \sum_{\gamma_1 \alpha_1 + \dots + \gamma_s \alpha_s = \mathrm{mdeg}(r)} c_\gamma \mathbf{y}^\gamma, \tag{3}$$

where the c_γ's are in Λ. Choose, for each $i = 1, \dots, s$, an element $x_i \in F_{\alpha_i}(R)$ such that $y_i = x_i + F^-_{\alpha_i}(R)$. Let M denote the $s \times n$ matrix whose rows are $\alpha_1, \dots, \alpha_s$. Write the equality (3) as

$$r + F^-_{\mathrm{mdeg}(r)}(R) = \sum_{\gamma M = \mathrm{mdeg}(r)} c_\gamma \mathbf{x}^\gamma + F^-_{\mathrm{mdeg}(r)}(R) \tag{4}$$

Therefore, we can prove by induction on $\mathrm{mdeg}(r)$ that

$$r = \sum_{\gamma M \preceq \mathrm{mdeg}(r)} a_\gamma \mathbf{x}^\gamma, \tag{5}$$

where $a_\gamma \in \Lambda$. To deduce that $\{\mathbf{x}^\gamma \mid \gamma \in \mathbb{N}^s\}$ is a basis for ${}_\Lambda R$ we only need to check the linear independence. Given a relation

$$\sum_{\gamma M \preceq \alpha} a_\gamma \mathbf{x}^\gamma = 0, \tag{6}$$

we proceed by induction on α. The relation (6) can be written as

$$\sum_{\gamma M = \alpha} a_\gamma \mathbf{x}^\gamma + \sum_{\gamma M \prec \alpha} a_\gamma \mathbf{x}^\gamma = 0 \tag{7}$$

which, in $G^F(R)$, gives

$$\sum_{\gamma M = \alpha} a_\gamma \mathbf{y}^\gamma = 0$$

As the monomials $\mathbf{y}^\gamma$ are Λ–linearly independent, we have that $a_\gamma = 0$ for $\gamma M = \alpha$. The remaining coefficients are zero by induction in view of (7).

Let $a \in \Lambda$ and $i \in \{1, \dots, s\}$. Since $G_0^F(R) = F_0(R) = \Lambda$ and $y_i a = \sigma_i(a) y_i$ we get $\sigma_i(a)$ has degree 0, i.e., $\sigma_i(a) \in \Lambda$. Write $a^{(i)} = \sigma_i(a)$. Then

$$0 = y_i a - a^{(i)} y_i = (x_i a - a^{(i)} x_i) + F^-_{\alpha_i}(R) \tag{8}$$

Since Λ is left noetherian and $F_{\alpha_i}(R)$ is finitely generated as a left Λ–module, we have that $F^-_{\alpha_i}(R)$ is a noetherian left Λ–module. Thus, we deduce from (8), in conjunction with (5), that

$$x_i a = a^{(i)} x_i + \sum_{\gamma \in \Gamma_i} a_\gamma \mathbf{x}^\gamma, \tag{9}$$

for some $a_\gamma \in \Lambda$, where Γ_i is a finite subset of $\mathbb{N}^s$ such that $\gamma M \prec \alpha_i$ for every $\gamma \in \Gamma_i$. On the other hand, for $1 \le i < j \le s$, we have

$$\begin{aligned} 0 &= y_j y_i - q_{ji} y_i y_j \\ &= (x_j + F^-_{\alpha_j}(R))(x_i + F^-_{\alpha_i}(R)) - q_{ji}(x_i + F^-_{\alpha_i}(R))(x_j + F^-_{\alpha_j}(R)) \\ &= (x_j x_i - q_{ji} x_i x_j) + F^-_{\alpha_i + \alpha_j}(R), \end{aligned}$$

which entails, by (5),

$$x_j x_i - q_{ji} x_i x_j = \sum_{\gamma \in \Gamma_{ij}} a_\gamma \mathbf{x}^\gamma, \tag{10}$$

where Γ_{ij} is a finite subset of $\mathbb{N}^s$ such that $\gamma M \prec \alpha_i + \alpha_j$ for every $\gamma \in \Gamma_{ij}$. Let $\preceq'$ be the admissible order on $\mathbb{N}^s$ defined by

$$\gamma \preceq' \mu \iff \begin{cases} \gamma M \prec \mu M & \text{or} \\ \gamma M = \mu M \quad \text{and} \quad \gamma \le_{\text{lex}} \mu \end{cases} \tag{11}$$

Since $\alpha_i = \epsilon_i M$ for every $i = 1, \ldots, s$, the relations (9) and (10) can be written as

$$x_i a - a^{(i)} x_i = \sum_{\substack{\gamma \prec' \epsilon_i \\ \gamma \in \Gamma_i}} a_\gamma \mathbf{x}^\gamma \tag{12}$$

and

$$x_j x_i - q_{ji} x_i x_j = \sum_{\substack{\gamma \prec' \epsilon_i + \epsilon_j \\ \gamma \in \Gamma_{ij}}} a_\gamma \mathbf{x}^\gamma, \tag{13}$$

which gives (iii).

(iii) implies (ii). First, notice that, by hypothesis, the relations (12) and (13) are satisfied. Let

$$C = \{0\} \cup \Big(\bigcup_{1 \le i \le s} C_i \Big) \cup \Big(\bigcup_{1 \le i < j \le s} C_{ij} \Big),$$

where $C_i = \Gamma_i - \epsilon_i$ and $C_{ij} = \Gamma_{ij} - \epsilon_i - \epsilon_j$. Clearly C is a finite subset of $\mathbb{Z}^s$ whose maximum with respect to $\preceq$ is 0. By [5, Corollary 2.2] (see also [15] and [17]), there is $\mathbf{w} = (w_1, \ldots, w_s) \in \mathbb{N}_+^s$ such that $\langle \mathbf{w}, \alpha \rangle < 0$ for every $\alpha \in C$. This implies that the relations (12) and (13) can be written as

$$x_i a - a^{(i)} x_i = \sum_{\langle \mathbf{w}, \gamma \rangle < w_i} a_\gamma \mathbf{x}^\gamma \tag{14}$$

and

$$x_j x_i - q_{ji} x_i x_j = \sum_{\langle \mathbf{w}, \gamma \rangle < w_i + w_j} a_\gamma \mathbf{x}^\gamma \tag{15}$$

By [5, Proposition 1.13], the family $\{H_\alpha(R) \mid \alpha \in \mathbb{N}^s\}$ where $H_\alpha(R)$ is the left Λ–module generated by the set $\{\mathbf{x}^\beta \mid \beta \preceq_{\mathbf{w}} \alpha\}$, is an $(\mathbb{N}^s, \preceq_{\mathbf{w}})$–filtration on R. Since $\mathbf{w}$ has no zero component, it follows that $H_\alpha(R)$ is finitely generated as a left Λ–module for every α. For each $n \in \mathbb{N}$, define $R_n = \bigcup_{\langle \mathbf{w}, \alpha \rangle \leqslant n} H_\alpha(R)$, which is a finitely generated left Λ–module. A straightforward verification shows that $\{R_n \mid n \in \mathbb{N}\}$ is a filtration on R. Clearly, $R_n = \sum_{\langle \mathbf{w}, \alpha \rangle \leqslant n} \Lambda \mathbf{x}^\alpha$ for every $\alpha \in \mathbb{N}^s$. Finally, let $y_i = x_i + R_{w_i - 1}$ for $1 \leqslant i < j \leqslant s$. By (14) and (15), $x_i a = a^{(i)} x_i$ for every $a \in \Lambda$ and $x_j x_i = q_{ji} x_i x_j$. Moreover, since the monomials $\mathbf{x}^\alpha$ are Λ–linearly independent, it follows that $\{\mathbf{y}^\alpha \mid \alpha \in \mathbb{N}^s\}$ is a left Λ–basis for $\mathrm{gr}(R)$. It follows from [11, 2.1.(iii)] that

$$\mathrm{gr}(R) \cong \Lambda[y_1; \sigma_1] \cdots [y_s; \sigma_s]$$

Finally, (ii) implies (i) obviously. □

In the following corollary, $K_0(R)$ denotes the Grothendieck group of R. Of course, the corollary says something new for rings satisfying (i) or (iii) in Theorem 1.

Corollary 2. *Assume R satisfies one (and then all) of the equivalent conditions of Theorem 1. Suppose, in addition, that Λ is right noetherian, q_{ji} is a unit of Λ for $1 \le i < j \le s$ and that σ_i is an automorphism of Λ for $i = 1, \ldots, s$.*

1. If every cyclic right Λ–module has finite projective dimension, then $K_0(\Lambda) \cong K_0(R)$.
2. If Λ is Auslander-regular then R Auslander-regular.

Proof. The first statement is a consequence of [12, Theorem 12.6.13]. If Λ is Auslander-regular, then, by [8, Theorem 4.2], $\mathrm{gr}(R) = \Lambda[y_1;\sigma_1]\cdots[y_s;\sigma_s]$ is Auslander-regular. The result follows now from [2, Theorem 3.9]. □

Theorem 3. *Assume that R is an algebra over a field $\mathbf{k}$ satisfying one (and then all) of the equivalent conditions of Theorem 1. Suppose, in addition, that*

(a) The scalars q_{ji} are units of $\mathbf{k}$ and the endomorphisms $\sigma_i : \Lambda \to \Lambda$ are automorphisms.
(b) Λ is generated as an algebra by elements $z_1, \ldots, z_t$ such that the standard filtration Λ_n obtained by giving degree 1 to each z_i satisfies that $gr(\Lambda) = \oplus_{n\geqslant 0}\Lambda_n/\Lambda_{n-1}$ is a finitely presented and noetherian algebra over $\mathbf{k}$.
(c) $\sigma_i(\Lambda_1) \subseteq \Lambda_1$, for $i = 1, \ldots, s$.
(d) either $gr(\Lambda)$ or $\Lambda[y_1;\sigma_1]\cdots[y_s;\sigma_s]$ is an Auslander-regular and Cohen-Macaulay algebra.

Then R is an Auslander-regular and Cohen-Macaulay algebra.

Proof. Let R_n be the filtration on R given by Theorem 1 with $\mathrm{gr}(R) = \Lambda[y_1;\sigma_1]\cdots[y_s;\sigma_s]$. Since $\sigma_i(\Lambda_1) \subseteq \Lambda_1$ for every $i = 1, \ldots, s$ and the filtration Λ_n is standard, we get that $y_i\Lambda_n \subseteq \Lambda_n y_i$ for every $i = 1, \ldots, s$ and every $n \geqslant 0$. Therefore, $\Lambda \subseteq \Lambda[y_1;\sigma_1]\cdots[y_s;\sigma_s]$ is a $\preceq_{\mathbf{w}}$–bounded extension of Λ in the sense of [5, Definition 1.8]. Here, $\mathbf{w} = (w_1, \ldots, w_s)$ with $w_i = \deg(y_i)$, $i = 1, \ldots, s$. Let $\preceq$ be the admissible order defined by

$$(i,\alpha) \preceq (j,\beta) \iff \begin{cases} \alpha \prec_{\mathbf{w}} \beta & \text{or} \\ \alpha = \beta \text{ and } i \leq j \end{cases}$$

Write $H(i,\alpha) = \sum_{(j,\beta)\preceq(i,\alpha)} \Lambda_j \mathbf{y}^\beta$. By [5, Proposition 1.13], these vector subspaces form a $(\mathbb{N}^{s+1}, \preceq)$–filtration for $\mathrm{gr}\, R$. Let $\mathrm{gr}(R)_{(n)} = \sum_{i+\langle \mathbf{w},\alpha\rangle \leq n} \Lambda_i \mathbf{y}^\alpha$. Since

$$\mathrm{gr}(R)_{(n)} = \bigcup_{\langle (1,\mathbf{w}),(i,\alpha)\rangle} H_{(i,\alpha)},$$

it follows that $\{\mathrm{gr}(R)_{(n)} \mid n \in \mathbb{N}\}$ is a filtration on $\mathrm{gr}(R)$. Moreover, the inclusion $\Lambda \subseteq \mathrm{gr}(R)$ is a strict filtered morphism, hence $\mathrm{gr}(\Lambda)$ can be viewed as a subalgebra of $\mathrm{gr}(\mathrm{gr}(R))$. Therefore, $\mathrm{gr}(\mathrm{gr}(R)) \cong \mathrm{gr}(\Lambda)[y_1;\sigma_1]\cdots[y_s;\sigma_s]$. Here, σ_i denotes the graded automorphism induced by the homonymous filtered automorphism of $\Lambda[y_1;\sigma_1]\cdots[y_{i-1};\sigma_{i-1}]$. Since $\mathrm{gr}(\Lambda)$ is a finitely presented and noetherian algebra, we see that $\mathrm{gr}(\mathrm{gr}(R))$ enjoys the same properties. Thus, the filtration R_n satisfies the hypotheses of [14, Theorem 1.3]. Now every finitely generated left R–module is endowed with a filtration such that $\mathrm{gr}(M)$ is finitely generated. By [14, Theorem 1.3], $\mathrm{GKdim}(M) = \mathrm{GKdim}(\mathrm{gr}(M))$. In particular, $\mathrm{GKdim}(R) = \mathrm{GKdim}(\mathrm{gr}(R))$. On the other hand, from the proof of [2, Theorem 3.9] we obtain that $j_R(M) = j_{\mathrm{gr}(R)}(\mathrm{gr}(M))$. If we assume that $\mathrm{gr}(R) = \Lambda[y_1;\sigma_1]\cdots[y_s;\sigma_s]$ is Cohen-Macaulay, then

$$\mathrm{GKdim}(R) = \mathrm{GKdim}(\mathrm{gr}(R)) = j_{\mathrm{gr}(R)}(\mathrm{gr}(M)) + \mathrm{GKdim}(\mathrm{gr}(M)) = j_R(M) + \mathrm{GKdim}(M),$$

whence R is Cohen-Macaulay too.

Lastly, if $\mathrm{gr}(\Lambda)$ is Cohen-Macaulay, then $\mathrm{gr}(\mathrm{gr}(R))$ satisfies the hypotheses of [16, Lemma], which implies that it is Cohen-Macaulay. Since the filtration $\mathrm{gr}(R)_{(n)}$ is finite-dimensional, we obtain that $\mathrm{gr}(R)$ is Cohen-Macaulay. Thus, R is Cohen-Macaulay by the foregoing argument. □

If $Q = (q_{ij})$ is a multiplicatively anti-symmetric $s \times s$ matrix with coefficients in $\mathbf{k}$, the coordinate ring of the quantum affine space $\mathcal{O}_Q(\mathbf{k}^s) = \mathbf{k}_Q[x_1, \ldots, x_s]$ is the $\mathbf{k}$–algebra generated by $x_1, \ldots, x_s$ subject to the relations $x_j x_i = q_{ji} x_i x_j$.

For our purposes, we are interested in certain localizations of $\mathcal{O}_Q(\mathbf{k}^s)$. Thus, consider some of the variables which, for simplicity, we assume to be $x_1, \ldots x_t$ with $t \leqslant s$. Since $x_1, \ldots, x_t$ are normal elements, they generate a multiplicatively closed Ore set, so that we can construct the localized algebra

$$\mathbf{k}_Q[x_1^{\pm 1}, \ldots, x_t^{\pm 1}, x_{t+1}, \ldots, x_s]$$

Although the following proposition should be well-known, we have not found a precise reference.

Proposition 4. *The algebra* $A = \mathbf{k}_Q[x_1^{\pm 1}, \ldots, x_t^{\pm 1}, x_{t+1}, \ldots, x_s]$ *is Auslander-regular and Cohen-Macaulay.*

Proof. Clearly, A is an iterated Ore extension of a McConnell-Pettit algebra, whence its global homological dimension is finite by [13, 3.1] and [8, Theorem 4.2]. On the other hand,

$$\mathbf{k}_Q[x_1, \ldots, x_t, x_{t+1}, \ldots, x_s]$$

is Auslander-regular and Cohen-Macaulay (see, e.g., [10, Theorem 3.5]). By [1, Proposition 2.1], A satisfies the Auslander condition. Since the multiplicative set generated by $x_1, \ldots, x_t$ consists of monomials, which are local normal elements, we have, by [1, Theorem 2.4], that our algebra A is Cohen-Macaulay. □

Theorem 5. *The quantized enveloping* $\mathbb{C}(q)$*–algebra* $U_q(C)$ *associated to a Cartan matrix* C *is Auslander-regular and Cohen-Macaulay.*

Proof. Accordingly with [7, Proposition 10.1], $U = U_q(C)$ is endowed with a $(\mathbb{N}^n, \preceq)$–filtration $\{F_\alpha(U) \mid \alpha \in \mathbb{N}^n\}$ for some n and a lexicographical order $\preceq$ in such a way that the multi-graded associated algebra $G^F(U) \cong \mathbb{C}(q)_Q[x_1^{\pm 1}, \ldots, x_t^{\pm 1}, x_{t+1}, \ldots, x_s]$ for a certain multiplicatively anti-symmetric matrix Q. By Proposition 4, $G^F(U)$ is Auslander-regular and Cohen-Macaulay. Moreover, $F_0(U) = \mathbb{C}(q)[z_1^{\pm 1}, \ldots, z_t^{\pm 1}]$, a commutative Laurent polynomial ring. Filter $F_0(U)$ with the standard filtration obtained by giving degree 1 to $z_i^{\pm 1}$ $(i = 1, \ldots, t)$. Then $gr(F_0(U))$ is a factor algebra of the commutative polynomial ring in $2t$ variables with coefficients in $\mathbb{C}(q)$. In particular, it is finitely presented and noetherian. Therefore, the hypotheses of Theorem 3 are fulfilled and, hence, $U_q(C)$ is Auslander-regular and Cohen-Macaulay. □

Remark 6. In [3, Proposition 2.2] it is shown that $U_q(C)$ is Auslander-regular. It is also proved [3, Theorem 2.3] that $U_q(C)$ is Cohen-Macaulay with respect to the Krull dimension in case q is a root of unity..

Remark 7. The normal separation of the prime spectrum of $U_q(C)$ would imply in view of Theorem 4 and [10, Theorem 1.6] that $U_q(C)$ is catenary. However, the (classical) universal enveloping algebras are not normally separated in general. So, as the referee pointed out, it is interesting to know if $U_q(C)$ does not really enjoy this property and why.

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